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Journal of Number Theory 123 (2007) 35–58
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

Transcendence of certain series involving binary linear recurrences

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Received 1 September 2004; revised 5 December 2005

Available online 5 July 2006

Communicated by David Goss

Abstract

Duverney and Nishioka [D. Duverney, Ku. Nishioka, An inductive method for proving the transcendence of certain series, Acta Arith. 110 (4) (2003) 305–330] studied the transcendence of $\sum_{k\geqslant 0}\frac{E_k(\alpha^{r^k})}{F_k(\alpha^{r^k})}$, where $E_k(z)$, $F_k(z)$ are polynomials, α is an algebraic number, and r is an integer greater than 1, using an inductive method. We extend their inductive method to the case of several variables. This enables us to prove the transcendence of $\sum'_{k\geqslant 0}\frac{a_k}{R_{cr^k+d}}$, where R_n is a binary linear recurrence and $\{a_k\}$ is a sequence of algebraic numbers.

MSC: 11J81

Keywords: Transcendence; Binary linear recurrence

1. Introduction

We consider a binary linear recursive sequence defined as follows: Let A_1, A_2 be integers. Let $\{R_n\}_{n\geqslant 0}$ satisfy

$$R_{n+2} = A_1R_{n+1} + A_2R_n, \tag{1}$$

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doi:10.1016/j.jnt.2006.05.019

where $R_0, R_1 \in \mathbb{Z}$ are not both zero. We put $P(X) = X^2 - A_1X - A_2$. Let ρ_1, ρ_2 be the roots of $P(X)$ with $|\rho_1| \geqslant |\rho_2|$. We assume $\Delta = A_1^2 + 4A_2 > 0$. Then ρ_1, ρ_2 are distinct real numbers.

Let r, c, and d be integers with $r \geqslant 2$ and $c \geqslant 1$. The arithmetic nature of the reciprocal sum

$$\sum_{k\geqslant 0}{}' \frac{a_k}{R_{cr^k+d}}, \tag{2}$$

where $\{a_k\}_{k\geqslant 0}$ is a sequence of algebraic numbers and the sum $\sum'_{k\geqslant 0}$ is taken over those k with $cr^k + d \geqslant 0$ and $R_{cr^k+d} \neq 0$, has been investigated by many authors. Let $\{F_n\}_{n\geqslant 0}$ be Fibonacci numbers defined by $F_{n+2} = F_{n+1} + F_n$ $(n \geqslant 0)$ with $F_0 = 0$, $F_1 = 1$. Lucas [12, p. 225] gave the explicit formula

$$\sum_{k\geqslant 0} \frac{1}{F_{2^k}} = \frac{7-\sqrt{5}}{2}.$$

Hoggatt and Bicknell [10] gave a more general formula

$$\sum_{k\geqslant 0} \frac{1}{F_{c2^k}} = \frac{1}{F_c} + \frac{\Phi+2}{\Phi(\Phi^{2c}-1)},$$

where $\Phi = \frac{1+\sqrt{5}}{2}$. In the special case where $\{R_n\}$ is of Fibonacci type $R_n = (\rho_1^n - \rho_2^n)/(\rho_1 - \rho_2)$ or of Lucas type $R_n = \rho_1^n + \rho_2^n$ or when one of the characteristic roots is ± 1, many authors have been studied the arithmetic nature of the reciprocal sum (2). For $a_k = (\pm 1)^k$, the irrationality [1,2,8] or even transcendence [15] of the sum (2) have been proved. For $a_k = \frac{1}{k!}$, Mignotte [14] and Mahler [13] proved the transcendence of the sum independently. For a more general $\{a_k\}$, Hančl and Kiss [9] and Bundschuh and Pethö [4] studied respectively the irrationality and the transcendence. In the general case of $\{R_n\}$, Becker and Töpfer [3] proved the transcendence of the sum (2) when $\{a_k\}$ is a periodic sequence of algebraic numbers and Δ is not a perfect square. Nishioka [17] established the transcendence for any linear recursive sequence $\{a_k\}$. In fact she proved the algebraic independence of the sums (2) for various d. The algebraic independence of the sums for various d and r has been studied in [18,19]. Tanaka [20] proved the algebraic independence of the following sums

$$\sum_{k\geqslant 0}{}' \frac{k^l \alpha^k}{(R_{a_k})^m} \quad \left(m \in \mathbb{N},\ l \in \mathbb{N} \cup \{0\},\ \alpha \in \overline{\mathbb{Q}}^{\times}\right),$$

where $\{a_k\}$ is a suitable linear recursive sequence. Duverney, Kanoko, and Tanaka [7] proved the transcendence of the sum

$$\sum_{k\geqslant 0}{}' \frac{a^k}{R_{cr^k} + b},$$

where $a \in \overline{\mathbb{Q}}^{\times}$, $b \in \mathbb{Z}$.

Recently Duverney and Nishioka [6] have made a great improvement for $\{a_k\}$. If α is an algebraic number, we denote its house by $\overline{|\alpha|} = \max\{|\alpha^\sigma| \mid \sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})\}$ and by $\mathrm{den}(\alpha)$ the least positive integer such that $\mathrm{den}(\alpha)\alpha$ is an algebraic integer, and we set $\|\alpha\| = \max\{\overline{|\alpha|},\ \mathrm{den}(\alpha)\}$.

Let $\boldsymbol{K}$ be an algebraic number field and $O_{\boldsymbol{K}}$ the ring of integers in $\boldsymbol{K}$. They considered the following function

$$\Phi_0(x) = \sum_{k \geqslant 0} \frac{E_k(x^{r^k})}{F_k(x^{r^k})},$$

where

$$\begin{aligned} &E_k(x) = a_{k1}x + a_{k2}x^2 + \cdots + a_{kL}x^L \in \boldsymbol{K}[x], \\ &F_k(x) = 1 + b_{k1}x + b_{k2}x^2 + \cdots + b_{kL}x^L \in O_{\boldsymbol{K}}[x], \\ &\log\|a_{kl}\|,\ \log\|b_{kl}\| = o\big(r^k\big), \quad 1 \leqslant l \leqslant L. \end{aligned}$$

Then they proved the following:

Transcendence Criterion. (Duverney and Nishioka [6]) *Let α be an algebraic number with $0 < |\alpha| < 1$ such that $F_k(\alpha^{r^k}) \neq 0$ for every $k \geqslant 0$, then $\Phi_0(\alpha)$ is an algebraic number if and only if $\Phi_0(x)$ is a rational function.*

As an application, they obtained necessary and sufficient conditions for transcendence of $\sum_{k\geqslant 0} \frac{a_k}{F_{r^k}+b_k}$ and $\sum_{k\geqslant 0} \frac{a_k}{L_{r^k}+b_k}$, where F_n is the nth Fibonacci number, L_n the nth Lucas number defined by $L_{n+2} = L_{n+1} + L_n$ $(n \geqslant 0)$ with $L_0 = 2, L_1 = 0$, and $\{a_k\}_{k\geqslant 0}$ and $\{b_k\}_{k\geqslant 0}$ are sequences in $\boldsymbol{K}$ and $O_{\boldsymbol{K}}$ respectively with $\log\|a_k\|, \log\|b_k\| = o(r^k)$.

In this paper, we will give necessary and sufficient conditions for the transcendence of the sums (2). We can express R_n $(n \geqslant 0)$ as

$$R_n = g_1\rho_1^n + g_2\rho_2^n, \quad |\rho_1| \geqslant |\rho_2|.$$

We note that $R_{cr^k+d} = 0$ for all large k if $R_{cr^k+d} = 0$ for infinitely many k.

Theorem 1. *Let $\{R_n\}$ be non-periodic and $R_{cr^k+d} \neq 0$ for infinitely many k. Let $\{a_k\}_{k\geqslant 0}$ be a sequence in $\boldsymbol{K}$ with $\log\|a_k\| = o(r^k)$. If there exist infinitely many k such that $a_k \neq 0$, then*

$$\theta = {\sum_{k\geqslant 0}}' \frac{a_k}{R_{cr^k+d}} \notin \overline{\mathbb{Q}}$$

except in the following two cases:

(1) *If $r = 2$, there exist $a \in \boldsymbol{K}$ and $N \in \mathbb{N}$ such that $a_n = a$ for every $n \geqslant N$, $|A_2| = 1$, and $g_1\rho_1^d + g_2\rho_2^d = 0$, then $\theta \in \boldsymbol{K}(\sqrt{\Delta})$.*
(2) *If $r = 2$, there exist $a \in \boldsymbol{K}$ and $N \in \mathbb{N}$ such that $a_n = a2^n$ for every $n \geqslant N$, $\rho_2 = \pm 1$, and $g_1\rho_1^d = g_2\rho_2^d$, then $\theta \in \boldsymbol{K}$.*

The binary linear recursive sequences can be classified into three types. I: $|A_2| = 1$. II: the characteristic roots ρ_1, ρ_2 are multiplicatively dependent and $|A_2| \geqslant 2$. III: the characteristic roots ρ_1, ρ_2 are multiplicatively independent (and then $|A_2| \geqslant 2$). The Transcendence Criterion of Duverney and Nishioka can almost cover the cases I and II. The important cases of $\{F_n\}$ and $\{L_n\}$ belong to the case I. They determined when $\Phi_0(x)$ is a rational function in the case $L \leqslant r$, which is enough to treat the case I. In studying the case II, we prove the irrationality of $\Phi_0(x)$ for $L > r$. For this, we will use all the absolute values on $\boldsymbol{K}$. For the case III, we have to introduce the following functions of several variables.

We use the usual notations

$$|\boldsymbol{\lambda}| = \sum_{i=1}^{m} \lambda_i, \qquad \boldsymbol{\alpha}^{\boldsymbol{\lambda}} = \prod_{i=1}^{m} {\alpha_i}^{\lambda_i}, \quad \text{and} \quad \langle \boldsymbol{\lambda}, \boldsymbol{\eta} \rangle = \sum_{i=1}^{m} \lambda_i \eta_i$$

for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)$. Let $r \geqslant 2$. We define $\Omega_n \boldsymbol{z} := (z_1^{r^n}, \ldots, z_m^{r^n})$ for $\boldsymbol{z} = (z_1, \ldots, z_m)$ and

$$S := \Phi_0(\boldsymbol{z}) = \sum_{k \geqslant 0} \frac{E_k(\Omega_k \boldsymbol{z})}{F_k(\Omega_k \boldsymbol{z})} \in \boldsymbol{K}[\![\boldsymbol{z}]\!] = \boldsymbol{K}[\![z_1, \ldots, z_m]\!], \tag{3}$$

where

$$E_k(\boldsymbol{z}) = \sum_{1 \leqslant |\boldsymbol{\lambda}| \leqslant L_E} a_{k\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}}, \qquad F_k(\boldsymbol{z}) = 1 + \sum_{1 \leqslant |\boldsymbol{\lambda}| \leqslant L_F} b_{k\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}} \in \boldsymbol{K}[\boldsymbol{z}]$$

with

$$\log \|a_{k\boldsymbol{\lambda}}\|, \log \|b_{k\boldsymbol{\lambda}}\| = o\big(r^k\big).$$

We define maximum total degrees of $E_k(\boldsymbol{z})$ and $F_k(\boldsymbol{z})$ as L_E and L_F, respectively. We consider the value of S at $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\boldsymbol{K}^{\times})^m$ with $0 < |\alpha_1|, \ldots, |\alpha_m| < 1$ such that $|\alpha_1|, \ldots, |\alpha_m|$ are multiplicatively independent and $F_k(\Omega_k \boldsymbol{\alpha}) \neq 0$ for every $k \geqslant 0$.

Theorem 2. *Let $S = \Phi_0(\boldsymbol{z})$ be defined by* (3)*, where*

$$E_k(\boldsymbol{z}) \in \boldsymbol{K}[z_1, \ldots, z_l], \quad F_k(\boldsymbol{z}) \in [z_{l+1}, \ldots, z_m] \quad (1 \leqslant l \leqslant m).$$

If $L_E < r$ and there exist infinitely many n such that $E_n(\boldsymbol{z}) \neq 0$, then $\Phi_0(\boldsymbol{\alpha})$ is a transcendental number.

Theorem 2 is not as good as the Transcendence Criterion of Duverney and Nishioka, however it is enough for our aim. To prove Theorem 2, the following theorem, which is a generalization of Theorem 2 in [6], plays an essential role. Here we define a symbol as

$$\operatorname{ord}\big(f(\boldsymbol{z})\big) = \min\big\{|\boldsymbol{\lambda}| \;\big|\; \sigma_{\boldsymbol{\lambda}} \neq 0\big\}$$

for $f(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}} \sigma_{\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}} \in \boldsymbol{K}[\![\boldsymbol{z}]\!]$ and $\operatorname{ord}(0) = \infty$ for convenience's sake.

Theorem 3. *Suppose there is a positive constant c_1 such that for any $M \geqslant 1$ and any $A_0, A_1 \in \boldsymbol{K}[z]$, not both zero, satisfying* $\deg A_0, \deg A_1 \leqslant M$,

$$\operatorname{ord}(A_0 + A_1 S) \leqslant c_1 M. \tag{4}$$

Then for any positive integer d there is a positive constant c_d such that for any $M \geqslant 1$ and any $A_0, \dots, A_d \in \boldsymbol{K}[z]$, not all zero, satisfying $\deg A_0, \dots, \deg A_d \leqslant M$,

$$\operatorname{ord}\bigl(A_0 + A_1 S + \cdots + A_d S^d\bigr) \leqslant c_d M. \tag{5}$$

We have another application.

Theorem 4. *Let $\rho_1, \dots, \rho_m$ be algebraic numbers such that $|\rho_1|, \dots, |\rho_m|$ are multiplicatively independent and $|\rho_1| > \max\{1, |\rho_2|, \dots, |\rho_m|\}$. Let $\{a_k\}_{k\geqslant 0}, \{b_{ik}\}_{k\geqslant 0}$ $(1 \leqslant i \leqslant m)$ be sequences in $\boldsymbol{K}$ with $\log\|a_k\|, \log\|b_{ik}\| = o(r^k)$ and $b_{1k} \neq 0$ for all large k. If $a_k \neq 0$ for infinitely many k, then*

$$\sum_{k\geqslant 0}{}' \frac{a_k}{b_{1k}\rho_1^{r^k} + \cdots + b_{mk}\rho_m^{r^k}} \notin \overline{\mathbb{Q}},$$

where the sum $\sum'_{k\geqslant 0}$ is taken over those k with $b_{1k}\rho_1^{r^k} + \cdots + b_{mk}\rho_m^{r^k} \neq 0$.

Becker and Töpfer [3] proved a similar theorem under the assumption that $\rho_1, \dots, \rho_m$ are multiplicatively independent, while they assume that $\{a_k\}$ is a periodic sequence and $\{b_{1k}\}, \dots, \{b_{mk}\}$ are constant sequences.

A concrete example of Theorems 1 and 4 are

$$\sum_{k\geqslant 0} \frac{a_k}{2^{r^k} + 3^{r^k}} \notin \overline{\mathbb{Q}},$$

where $a_k \in \mathbb{Z}$ $(k \geqslant 0)$ with $\log|a_k| = o(r^k)$ and $a_k \neq 0$ infinitely many k. The denominator of this series is R_{r^n} satisfying $R_{n+2} = 5R_{n+1} - 6R_n$ $(n \geqslant 0)$ and $R_0 = 2$, $R_1 = 5$.

We note that we have

$$|\alpha| \geqslant \|\alpha\|^{-2[\mathbb{Q}(\alpha):\mathbb{Q}]}, \qquad \bigl\|\alpha^{-1}\bigr\| \leqslant \|\alpha\|^{2[\mathbb{Q}(\alpha):\mathbb{Q}]} \tag{6}$$

for non-zero algebraic α (cf. [16, Lemma 2.10.2]). For $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$, we can easily see that

$$\left\|\sum_{i=1}^n \alpha_i\right\| \leqslant n\prod_{i=1}^n \|\alpha_i\|, \qquad \left\|\prod_{i=1}^n \alpha_i\right\| \leqslant \prod_{i=1}^n \|\alpha_i\|. \tag{7}$$

2. Proof of Theorem 3

Put $L = \max\{L_E, L_F\}$ and

$$\Phi_n(z) = \sum_{k\geqslant 0} \frac{E_{n+k}(\Omega_k z)}{F_{n+k}(\Omega_k z)} \quad (n \geqslant 0).$$

Then we have

$$\Phi_n(\Omega_n z) = \sum_{k\geqslant 0} \frac{E_{n+k}(\Omega_{n+k}z)}{F_{n+k}(\Omega_{n+k}z)} = \sum_{k\geqslant n} \frac{E_k(\Omega_k z)}{F_k(\Omega_k z)}$$
$$= S - T_n(z), \tag{8}$$

where $T_n(z) = \sum_{k=0}^{n-1} \frac{E_k(\Omega_k z)}{F_k(\Omega_k z)} \in \boldsymbol{K}(z)$.

Proof of Theorem 3. We prove (5) by induction on d. If $d = 1$, (5) is the same as (4). Suppose that for a given $d \geqslant 2$,

$$\operatorname{ord}\bigl(B_0 + B_1 S + \cdots + B_{d-1} S^{d-1}\bigr) \leqslant c_{d-1} M \tag{9}$$

holds for any $B_0, \ldots, B_{d-1} \in \boldsymbol{K}[z]$, not all zero, with $\deg B_i \leqslant M$ $(0 \leqslant i \leqslant d-1)$. We may assume that $c_1, c_{d-1} > 1$ and $A_d \neq 0$. Let c be sufficiently large integer. (Precisely we can take $c = dm(dL)^m$.) Then there exist $Q_n(z) \neq 0$, $P_{ni}(z) \in \boldsymbol{K}[z]$ with $\deg Q_n(z), \deg P_{ni}(z) \leqslant c$ $(1 \leqslant i \leqslant d)$ such that

$$Q_n(z)\Phi_n(z)^i - P_{ni}(z) = \sum_{l=c+dL+1}^{\infty} A_{nil}(z) = G_{ni}(z) \quad (1 \leqslant i \leqslant d), \tag{10}$$

where $G_{ni}(z) \in \boldsymbol{K}[\![z]\!]$ and $A_{nil}(z) \in \boldsymbol{K}[z]$ are homogeneous polynomials of degree l. For this, first we choose $Q_n(z)$ in such a way that the terms of degrees $c+1, \ldots, c+dL$ vanish in the Taylor expansion of $Q_n(z)\Phi_n(z)^i$ $(1 \leqslant i \leqslant d)$. We have only to solve a linear homogeneous system which has $d\sum_{k=1}^{dL} {}_{c+k+m-1}C_{m-1}$ equations and $\sum_{k=0}^{c} {}_{k+m-1}C_{m-1}$ unknowns since the number of monomials in m variables of degree n is ${}_{n+m-1}C_{m-1}$. Since c is sufficient large, we have

$$d\sum_{k=1}^{dL} {}_{c+k+m-1}C_{m-1} = d\sum_{k=1}^{dL} \frac{(c+k+m-1)\cdots(c+k+1)}{(m-1)!}$$
$$= O\bigl(c^{m-1}\bigr)$$
$$< \sum_{k=0}^{c} \frac{(k+m-1)\cdots(k+1)}{(m-1)!}.$$

Hence the system has a non-trivial solution. Next we make the terms of degrees $0, \ldots, c$ vanish, using $P_{ni}(z)$ $(1 \leqslant i \leqslant d)$.

First, we show the inequality

$$c + dL + 1 \leqslant \operatorname{ord}\bigl(G_{n1}(z)\bigr) \leqslant c_1(c+L) = c_0. \tag{11}$$

Replacing z by $\Omega_n z$ in (10) and using (8), we have

$$Q_n(\Omega_n z)(S - T_n) - P_{n1}(\Omega_n z) = G_{n1}(\Omega_n z).$$

Multiplying both sides by $D_n = \prod_{k=0}^{n-1} F_k(\Omega_k z)$, we have

$$D_n Q_n(\Omega_n z)S - D_n Q_n(\Omega_n z)T_n - D_n P_{n1}(\Omega_n z) = D_n G_{n1}(\Omega_n z).$$

Since

$$\deg D_n,\ \deg D_n T_n \leqslant Lr^n, \tag{12}$$

we see that

$$\deg D_n Q_n(\Omega_n z),\ \ \deg D_n Q_n(\Omega_n z)T_n,\ \ \deg D_n P_{n1}(\Omega_n z) \leqslant (c+L)r^n.$$

By (4) and noting that ord $D_n = 0$, we have

$$(c + dL + 1)r^n \leqslant \operatorname{ord}\bigl(G_{n1}(\Omega_n z)\bigr) \leqslant c_1(c+L)r^n = c_0 r^n,$$

which implies (11).

We define $P_{n0}(z) = Q_n(z)$, $G_{n0}(z) = 0$. Then we have

$$Q_n(\Omega_n z)(S - T_n)^i - P_{ni}(\Omega_n z) = G_{ni}(\Omega_n z) \quad (0 \leqslant i \leqslant d), \tag{13}$$

or in matrix form

$$Q_n(\Omega_n z)M_n \begin{pmatrix} 1 \\ S \\ \vdots \\ S^d \end{pmatrix} - \begin{pmatrix} P_{n0}(\Omega_n z) \\ P_{n1}(\Omega_n z) \\ \vdots \\ P_{nd}(\Omega_n z) \end{pmatrix} = \begin{pmatrix} 0 \\ G_{n1}(\Omega_n z) \\ \vdots \\ G_{nd}(\Omega_n z) \end{pmatrix}, \tag{14}$$

where

$$M_n = \begin{pmatrix} 1 & & & & 0 \\ -T_n & 1 & & & \\ T_n^2 & -2T_n & 1 & & \\ \vdots & & & \ddots & \\ (-1)^d T_n^d & (-1)^{d-1}\binom{d}{1}T_n^{d-1} & \dots & \dots & 1 \end{pmatrix}.$$

In [5], it is shown that

$$M_n^{-1} = \begin{pmatrix} 1 & & & & 0 \\ T_n & 1 & & & \\ T_n^2 & 2T_n & 1 & & \\ \vdots & & & \ddots & \\ T_n^d & \binom{d}{1}T_n^{d-1} & \dots & \dots & 1 \end{pmatrix}.$$

Note that $D_n^d M_n^{-1}$ has its elements in $\boldsymbol{K}[z]$. Multiplying (14) on the left by M_n^{-1}, we get

$$Q_n(\Omega_n z)\begin{pmatrix} 1 \\ S \\ \vdots \\ S^d \end{pmatrix} - M_n^{-1}\begin{pmatrix} P_{n0}(\Omega_n z) \\ P_{n1}(\Omega_n z) \\ \vdots \\ P_{nd}(\Omega_n z) \end{pmatrix} = M_n^{-1}\begin{pmatrix} 0 \\ G_{n1}(\Omega_n z) \\ \vdots \\ G_{nd}(\Omega_n z) \end{pmatrix}. \tag{15}$$

For $A_0, \ldots, A_d \in \boldsymbol{K}[z]$, not all zero, satisfying $\deg A_0, \ldots, \deg A_d < M$, multiplying (15) on the left by the row matrix $D_n^d(A_0, \ldots, A_d)$, we obtain

$$U_n \sum_{h=0}^{d} A_h S^h - V_n = W_n, \tag{16}$$

where

$$U_n = D_n^d Q_n(\Omega_n z) \in \boldsymbol{K}[z] \setminus \{0\},$$

$$V_n = (A_0, \ldots, A_d) D_n^d M_n^{-1}\begin{pmatrix} P_{n0}(\Omega_n z) \\ P_{n1}(\Omega_n z) \\ \vdots \\ P_{nd}(\Omega_n z) \end{pmatrix} \in \boldsymbol{K}[z],$$

and

$$W_n = (A_0, \ldots, A_d) D_n^d M_n^{-1}\begin{pmatrix} 0 \\ G_{n1}(\Omega_n z) \\ \vdots \\ G_{nd}(\Omega_n z) \end{pmatrix} \in \boldsymbol{K}[\![z]\!].$$

Let n be the positive integer satisfying

$$r^{n-1} \leqslant c_{d-1} M < r^n. \tag{17}$$

Nothing that $D_n^d T_n^i = (D_n T_n)^i D_n^{d-i}$ $(0 \leqslant i \leqslant d)$ and (12), then

$$\begin{aligned} \deg V_n &\leqslant M + dLr^n + cr^n \\ &< r^n + dLr^n + cr^n \\ &= (c + dL + 1) r^n. \end{aligned} \tag{18}$$

By (11), there exists minimum p with $c + dL + 1 \leqslant p \leqslant c_0$ satisfying

$$\begin{pmatrix} 0 \\ A_{n1p}(z) \\ \vdots \\ A_{ndp}(z) \end{pmatrix} \neq 0.$$

So there exists $i \geqslant 1$ with

$$\begin{pmatrix} 0 \\ A_{n1p}(z) \\ \vdots \\ A_{ndp}(z) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_{nip}(z) \\ \vdots \\ A_{ndp}(z) \end{pmatrix}, \quad A_{nip}(z) \neq 0.$$

We put $I = \{f(z) \in \boldsymbol{K}[\![z]\!] \mid \operatorname{ord} f(z) \geqslant (p+1)r^n\}$. The set I also includes 0. Noting that $\operatorname{ord}(S - T_n) \geqslant r^n$ and $\operatorname{ord}(A_{njp}(\Omega_n z)) \geqslant pr^n$ $(i \leqslant j \leqslant d)$, we have, mod I,

$$\begin{aligned}
W_n &\equiv D_n^d(A_0, \dots, A_d) M_n^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_{nip}(\Omega_n z) \\ \vdots \\ A_{ndp}(\Omega_n z) \end{pmatrix} \\
&\equiv D_n^d(A_0, \dots, A_d) \begin{pmatrix} 1 & & & & 0 \\ T_n & 1 & & & \\ T_n^2 & 2T_n & 1 & & \\ \vdots & & & \ddots & \\ T_n^d & \binom{d}{1} T_n^{d-1} & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_{nip}(\Omega_n z) \\ \vdots \\ A_{ndp}(\Omega_n z) \end{pmatrix} \\
&\equiv D_n^d(A_0, \dots, A_d) \begin{pmatrix} 0 & & \dots & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & & \dots & \dots & 0 \\ 1 & & & & 0 \\ \binom{i+1}{i} T_n & & 1 & & \\ \vdots & & & \ddots & \\ \binom{d}{i} T_n^{d-i} & \binom{d}{i+1} T_n^{d-(i+1)} & & \dots & 1 \end{pmatrix} \begin{pmatrix} A_{nip}(\Omega_n z) \\ \vdots \\ A_{ndp}(\Omega_n z) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&\equiv D_n^d(A_0,\dots,A_d)\begin{pmatrix} 0 & \dots & \dots & 0\\ \vdots & & & \vdots\\ 0 & \dots & \dots & 0\\ 1 & & & 0\\ \binom{i+1}{i}S & 1 & & \\ \vdots & & \ddots & \\ \binom{d}{i}S^{d-i} & \binom{d}{i+1}S^{d-(i+1)} & \dots & 1\end{pmatrix}\begin{pmatrix} A_{nip}(\Omega_n z)\\ \vdots \\ A_{ndp}(\Omega_n z)\end{pmatrix}\\
&\equiv D_n^d\left(\sum_{|\boldsymbol{\lambda}|=p}\left(\sum_{h=0}^{d-i} B_{h\boldsymbol{\lambda}}S^h\right)(\Omega_n z)^{\boldsymbol{\lambda}}\right),
\end{aligned}
$$

where $B_{h\boldsymbol{\lambda}} \in \boldsymbol{K}[z]$, $\deg B_{h\boldsymbol{\lambda}} \leqslant M$. The coefficient of S^{d-i} is

$$D_n^d \sum_{|\boldsymbol{\lambda}|=p} B_{d-i,\boldsymbol{\lambda}}(\Omega_n z)^{\boldsymbol{\lambda}} = D_n^d A_d\binom{d}{i} A_{nip}(\Omega_n z) \neq 0.$$

Therefore there exists $\boldsymbol{\lambda}_0$ such that $|\boldsymbol{\lambda}_0| = p$ and $B_{d-i,\boldsymbol{\lambda}_0} \neq 0$. By induction hypothesis (9) and (17),

$$\operatorname{ord}\left(\sum_{h=0}^{d-i} B_{h\boldsymbol{\lambda}_0}S^h\right) \leqslant c_{d-1}M < r^n.$$

Therefore, for $\boldsymbol{\lambda}_0$ satisfying with $|\boldsymbol{\lambda}_0| = p$, we have

$$\operatorname{ord}\left(\left(\sum_{h=0}^{d-i} B_{h\boldsymbol{\lambda}_0}S^h\right)(\Omega_n z)^{\boldsymbol{\lambda}_0}\right) < (p+1)r^n.$$

Then we can see

$$\operatorname{ord}\left(\sum_{|\boldsymbol{\lambda}|=p}\left(\sum_{h=0}^{d-i} B_{h\boldsymbol{\lambda}}S^h\right)(\Omega_n z)^{\boldsymbol{\lambda}}\right) < (p+1)r^n.$$

In fact, if $|\boldsymbol{\mu}|$, $|\boldsymbol{\mu}'| < r^n$, $|\boldsymbol{\lambda}| = |\boldsymbol{\lambda}'| = p$, and $z^{\boldsymbol{\mu}}(\Omega_n z)^{\boldsymbol{\lambda}} = z^{\boldsymbol{\mu}'}(\Omega_n z)^{\boldsymbol{\lambda}'}$, we easily see $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, $\boldsymbol{\mu} = \boldsymbol{\mu}'$. Hence we get $W_n \not\equiv 0 \pmod{I}$. Therefore

$$(c + dL + 1)r^n \leqslant \operatorname{ord} W_n < (p+1)r^n \leqslant (c_0+1)r^n.$$

Suppose that $V_n \neq 0$. Then we have by (18)

$$\operatorname{ord} V_n \leqslant \deg V_n < (c + dL + 1)r^n \leqslant \operatorname{ord} W_n.$$

Hence we get

$$\begin{aligned}\operatorname{ord}\left(\sum_{h=0}^{d} A_h S^h\right) &\leqslant \operatorname{ord}\left(U_n \sum_{h=0}^{d} A_h S^h\right)\\ &= \operatorname{ord} V_n\\ &< (c+dL+1)r^n\\ &\leqslant (c+dL+1)rc_{d-1}M.\end{aligned}$$

If $V_n = 0$, we have

$$\begin{aligned}\operatorname{ord}\left(\sum_{h=0}^{d} A_h S^h\right) &\leqslant \operatorname{ord}\left(U_n \sum_{h=0}^{d} A_h S^h\right)\\ &= \operatorname{ord} W_n\\ &< (c_0+1)r^n\\ &\leqslant (c_0+1)rc_{d-1}M.\end{aligned}$$

Letting $c_d = (c+dL+c_0+1)rc_{d-1}$, we obtain (5). $\square$

3. Lemmas

Let $c_0, c_1, \ldots$ be positive constants and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 5. ([11], see [16, Theorem 2.9.1].) *If a sequence* $\{f_n(z)\}_{n\geqslant 0}$ *in* $\boldsymbol{K}[\![z]\!]$ *and* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\boldsymbol{K}^{\times})^m$ *satisfy the following conditions* (I)–(V), *then* $f_0(\boldsymbol{\alpha}) \notin \overline{\mathbb{Q}}$.

For each positive integer n, let Ω_n *be an* $m \times m$ *matrix with non-negative integer entries. We define* $\Omega_n z := (\prod_{j=1}^{m} z_j^{\omega_{1j}^{(n)}}, \ldots, \prod_{j=1}^{m} z_j^{\omega_{mj}^{(n)}})$, *where* $(\Omega_n)_{ij} = \omega_{ij}^{(n)}$ $(1 \leqslant i, j \leqslant m)$.

(I) $\omega_{ij}^{(n)} \leqslant c_1 r_n$, *where* $r_n > 0$ $(1 \leqslant i, j \leqslant m)$ *and* $\lim_{n\to\infty} r_n = \infty$.

Let $\Omega_n \boldsymbol{\alpha} = (\alpha_1^{(n)}, \ldots, \alpha_m^{(n)})$.

(II) $\log|\alpha_i^{(n)}| \sim -\eta_i r_n$ *as* $n \to \infty$ $(1 \leqslant i \leqslant m)$, *where* $\eta_1, \ldots, \eta_m$ *are positive and linearly independent over* $\mathbb{Q}$.

(III) $f_n(\Omega_n \boldsymbol{\alpha}) = a_n f_0(\boldsymbol{\alpha}) + b_n$ $(n \geqslant 1)$, *where* $a_n, b_n \in \boldsymbol{K}$ *and* $\log\|a_n\|, \log\|b_n\| \leqslant c_2 r_n$.

(IV) *Put* $f_n(z) = \sum_{\boldsymbol{\lambda}} \sigma_{\boldsymbol{\lambda}}^{(n)} z^{\boldsymbol{\lambda}}$ $(n \geqslant 0)$. *Then for any* $\varepsilon > 0$ *there is a positive constant* $c_3(\varepsilon)$ *such that* $\log\|\sigma_{\boldsymbol{\lambda}}^{(n)}\| \leqslant \varepsilon r_n(1+|\boldsymbol{\lambda}|)$ *for any* $n \geqslant c_3(\varepsilon)$ *and any* $\boldsymbol{\lambda} \in {\mathbb{N}_0}^m$.

Let $s_{\boldsymbol{\lambda}}$ $(\boldsymbol{\lambda} \in {\mathbb{N}_0}^m)$ *be variables and* $F(z; s) = F(z; \{s_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}}) = \sum_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}} z^{\boldsymbol{\lambda}}$. *Then* $F(z; \sigma^{(n)}) = F(z; \{\sigma_{\boldsymbol{\lambda}}^{(n)}\}_{\boldsymbol{\lambda}}) = f_n(z)$ $(n \geqslant 0)$.

(V) *If* $P_0(z; s), \ldots, P_d(z; s)$ *are polynomials in* $z_1, \ldots, z_m, \{s_{\boldsymbol{\lambda}}\}$ *with coefficients in* $\boldsymbol{K}$ *and* $E(z; s) = \sum_{j=0}^{d} P_j(z; s) F(z; s)^j$, *there is a positive integer* I *with the following prop-*

erty: *If n is sufficiently large and $P_0(z;\sigma^{(n)}),\dots,P_d(z;\sigma^{(n)})$ are not all zero, then* $\operatorname{ord}(E(z;\sigma^{(n)}))\leqslant I$.

We put

$$\Omega_n=\begin{pmatrix} r^n & & 0\\ & \ddots & \\ 0 & & r^n\end{pmatrix}$$

and set

$$\Phi_n(z)=\sum_{k\geqslant 0}\frac{E_{n+k}(\Omega_k z)}{F_{n+k}(\Omega_k z)}=\sum_{\boldsymbol\lambda}\sigma_{\boldsymbol\lambda}^{(n)}z^{\boldsymbol\lambda}, \tag{19}$$

where

$$E_k(z)=\sum_{1\leqslant|\boldsymbol\lambda|\leqslant L_E}a_{k\boldsymbol\lambda}z^{\boldsymbol\lambda},\qquad F_k(z)=1+\sum_{1\leqslant|\boldsymbol\lambda|\leqslant L_F}b_{k\boldsymbol\lambda}z^{\boldsymbol\lambda}\in\boldsymbol{K}[z]$$

with

$$\log\|a_{k\boldsymbol\lambda}\|,\ \log\|b_{k\boldsymbol\lambda}\|=o\big(r^k\big). \tag{20}$$

We consider the value of $\Phi_0(z)$ at $\boldsymbol\alpha=(\alpha_1,\dots,\alpha_m)\in(\boldsymbol{K}^\times)^m$ with $0<|\alpha_1|,\dots,|\alpha_m|<1$ such that $|\alpha_1|,\dots,|\alpha_m|$ are multiplicatively independent and $F_k(\Omega_k\boldsymbol\alpha)\neq 0$ for every $k\geqslant 0$. We will apply Lemma 5 to $\{\Phi_n(z)\}$ and $\boldsymbol\alpha$. Put $f_n(z)=\Phi_n(z)$. Then $r_n=r^n$ satisfies (I). Put $\eta_i=-\log|\alpha_i|>0$ $(1\leqslant i\leqslant m)$. Since $|\alpha_1|,\dots,|\alpha_m|$ are multiplicatively independent, $\eta_1,\dots,\eta_m$ are linearly independent over $\mathbb{Q}$, which means (II).

Lemma 6. *Let $T_n(z)$ be defined by* (3) *and* (8), *then*

$$\log\big\|T_n(\boldsymbol\alpha)\big\|=O\big(r^n\big).$$

Therefore $\{\Phi_n(z)\}_{n\geqslant 0}$ satisfies the property (III).

Proof. It follows from (6) and (7) that

$$\big\|T_n(\boldsymbol\alpha)\big\|\leqslant n\prod_{k=0}^{n-1}\big\|E_k(\Omega_k\boldsymbol\alpha)\big\|\big\|F_k(\Omega_k\boldsymbol\alpha)\big\|^{2[\boldsymbol{K}:\mathbb{Q}]}.$$

Putting $b_{k0}=1$, we have

$$\begin{aligned}\overline{\big|F_k(\Omega_k\boldsymbol\alpha)\big|}&\leqslant\sum_{|\boldsymbol\lambda|\leqslant L_F}\overline{|b_{k\boldsymbol\lambda}|}\|\alpha_1\|^{r^kL}\cdots\|\alpha_m\|^{r^kL}\\&\leqslant(L+1)^m\prod_{|\boldsymbol\lambda|\leqslant L_F}\|b_{k\boldsymbol\lambda}\|\prod_{j=1}^m\|\alpha_j\|^{r^kL},\end{aligned}$$

and

$$\operatorname{den}\big(F_k(\Omega_k \boldsymbol{\alpha})\big) \leqslant \prod_{|\boldsymbol{\lambda}| \leqslant L_F} \|b_{k\boldsymbol{\lambda}}\| \prod_{j=1}^{m} \|\alpha_j\|^{r^k L}.$$

Hence we get by (20)

$$\begin{aligned}
&\log\big\|F_k(\Omega_k \boldsymbol{\alpha})\big\| \\
&\quad \leqslant m\log(L+1) + \sum_{|\boldsymbol{\lambda}| \leqslant L_F} \log\|b_{k\boldsymbol{\lambda}}\| + r^k L \sum_{j=1}^{m} \log\|\alpha_j\| = O\big(r^k\big).
\end{aligned}$$

In the same way, we have $\log\|E_k(\Omega_k \boldsymbol{\alpha})\| = O(r^k)$ and therefore

$$\log\big\|T_n(\boldsymbol{\alpha})\big\| = O\big(r^n\big). \qquad \square$$

Lemma 7. *Let $\sigma_{\boldsymbol{\lambda}}^{(n)}$ be defined by* (19). *We assume either that*

$$E_k(z) \in \boldsymbol{K}[z_1, \ldots, z_l], \quad F_k(z) \in \boldsymbol{K}[z_{l+1}, \ldots, z_m] \quad (1 \leqslant l \leqslant m), \tag{21}$$

or that there is a positive integer D such that

$$DF_k(z) \in O_{\boldsymbol{K}}[z] \tag{22}$$

for all k. For any $\kappa > 1$, if n is sufficiently large,

$$\big\|\sigma_{\boldsymbol{\lambda}}^{(n)}\big\| \leqslant \kappa^{|\boldsymbol{\lambda}| r^n}.$$

Therefore $\{\Phi_n(z)\}_{n \geqslant 0}$ satisfies the property (IV).

Proof. By the assumption (20), we have $\|a_{n\boldsymbol{\lambda}}\| \leqslant \kappa^{r^n}$, $\|b_{n\boldsymbol{\lambda}}\| \leqslant \kappa^{r^n}$ for large n. First we estimate $|\overline{\sigma_{\boldsymbol{\lambda}}^{(n)}}|$. In fact we can estimate without conditions (21), (22). Let $\sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}} z^{\boldsymbol{\lambda}} \ll \sum_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}} z^{\boldsymbol{\lambda}}$ mean $|a_{\boldsymbol{\lambda}}| \leqslant b_{\boldsymbol{\lambda}}$ for all $\boldsymbol{\lambda}$. For large k, we have

$$E_k(z) \ll \kappa^{r^k} \sum_{1 \leqslant |\boldsymbol{\lambda}| \leqslant L} z^{\boldsymbol{\lambda}},$$

$$\frac{1}{F_k(z)} \ll 1 + \kappa^{r^k} \sum_{1 \leqslant |\boldsymbol{\lambda}| \leqslant L} z^{\boldsymbol{\lambda}} + \kappa^{2r^k} \Bigg(\sum_{1 \leqslant |\boldsymbol{\lambda}| \leqslant L} z^{\boldsymbol{\lambda}}\Bigg)^2 + \cdots.$$

Since $(\sum_{1 \leqslant |\boldsymbol{\lambda}| \leqslant L} z^{\boldsymbol{\lambda}})^q \ll (L+1)^{mq} (\sum_{q \leqslant |\boldsymbol{\lambda}|} z^{\boldsymbol{\lambda}})$, we get

$$\begin{aligned}
\frac{E_k(z)}{F_k(z)} &\ll \kappa^{r^k}(L+1)^m \sum_{1\leqslant|\boldsymbol{\lambda}|} z^{\boldsymbol{\lambda}} + \cdots \\
&\quad + \kappa^{qr^k}(L+1)^{qm} \sum_{q\leqslant|\boldsymbol{\lambda}|} z^{\boldsymbol{\lambda}} + \cdots \\
&\ll \sum_{1\leqslant|\boldsymbol{\lambda}|} \kappa^{|\boldsymbol{\lambda}|r^k}(L+1)^{m|\boldsymbol{\lambda}|} z^{\boldsymbol{\lambda}} + \cdots \\
&\quad + \sum_{q\leqslant|\boldsymbol{\lambda}|} \kappa^{|\boldsymbol{\lambda}|r^k}(L+1)^{m|\boldsymbol{\lambda}|} z^{\boldsymbol{\lambda}} + \cdots \\
&\ll \sum_{1\leqslant|\boldsymbol{\lambda}|} |\boldsymbol{\lambda}|\kappa^{|\boldsymbol{\lambda}|r^k}(L+1)^{m|\boldsymbol{\lambda}|} z^{\boldsymbol{\lambda}} \\
&\ll \sum_{1\leqslant|\boldsymbol{\lambda}|} \kappa^{2|\boldsymbol{\lambda}|r^k} z^{\boldsymbol{\lambda}}.
\end{aligned}$$

So we obtain

$$\begin{aligned}
\Phi_n(z) &= \sum_{k\geqslant 0} \frac{E_{n+k}(\Omega_k z)}{F_{n+k}(\Omega_k z)} \\
&\ll \sum_{k\geqslant 0} \sum_{1\leqslant|\boldsymbol{\mu}|} \kappa^{2|\boldsymbol{\mu}|r^{n+k}} (\Omega_k z)^{\boldsymbol{\mu}} \\
&\ll \sum_{1\leqslant|\boldsymbol{\lambda}|} \sum_{k=0}^{[\log_r|\boldsymbol{\lambda}|]} \kappa^{2|\boldsymbol{\lambda}|r^n} z^{\boldsymbol{\lambda}} \\
&= \sum_{1\leqslant|\boldsymbol{\lambda}|} \bigl(1+\bigl[\log_r|\boldsymbol{\lambda}|\bigr]\bigr)\kappa^{2|\boldsymbol{\lambda}|r^n} z^{\boldsymbol{\lambda}} \\
&\ll \sum_{1\leqslant|\boldsymbol{\lambda}|} \kappa^{3|\boldsymbol{\lambda}|r^n} z^{\boldsymbol{\lambda}}
\end{aligned}$$

if n is large. Hence we get $|\sigma_{\boldsymbol{\lambda}}^{(n)}| \leqslant (\kappa^3)^{|\boldsymbol{\lambda}|r^n}$. In the same way, we can get $\overline{|\sigma_{\boldsymbol{\lambda}}^{(n)}|} \leqslant (\kappa^3)^{|\boldsymbol{\lambda}|r^n}$.

Next we estimate $\operatorname{den}(\sigma_{\boldsymbol{\lambda}}^{(n)})$. We put

$$A_k = \prod_{1\leqslant|\boldsymbol{\lambda}|\leqslant L_F} \operatorname{den}(a_{k\boldsymbol{\lambda}}), \qquad B_k = \prod_{1\leqslant|\boldsymbol{\lambda}|} \operatorname{den}(b_{k\boldsymbol{\lambda}}).$$

If k is sufficiently large, we have $A_k, B_k \leqslant \kappa^{r^k}$.

Let

$$\begin{aligned}
F_k(z)^{-1} &= 1 + \bigl(1-F_k(z)\bigr) + \bigl(1-F_k(z)\bigr)^2 + \cdots \\
&= \sum_{\boldsymbol{\lambda}} c_{k\boldsymbol{\lambda}} z^{\boldsymbol{\lambda}}.
\end{aligned}$$

Then ${B_k}^{|\boldsymbol{\lambda}|} c_{k\boldsymbol{\lambda}} \in O_{\boldsymbol{K}}$ and

$$\begin{aligned}\Phi_n(z) &= \sum_{k\geqslant 0} E_{n+k}(\Omega_k z) F_{n+k}(\Omega_k z)^{-1} \\ &= \sum_{k\geqslant 0}\Bigg(\sum_{1\leqslant|\boldsymbol{\mu}_1|\leqslant L_E} a_{n+k,\boldsymbol{\mu}_1} z^{\boldsymbol{\mu}_1 r^k}\Bigg)\Bigg(\sum_{0\leqslant|\boldsymbol{\mu}_2|} c_{n+k,\boldsymbol{\mu}_2} z^{\boldsymbol{\mu}_2 r^k}\Bigg).\end{aligned}$$

First we assume (21). Let $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$, where $\boldsymbol{\lambda}_1 = (\lambda_1, \ldots, \lambda_l, 0, \ldots, 0)$, $\boldsymbol{\lambda}_2 = (0, \ldots, 0, \lambda_{l+1}, \ldots, \lambda_m)$. Then we see that

$$\sigma_{\boldsymbol{\lambda}}^{(n)} = \sum a_{n+k,\boldsymbol{\mu}_1} c_{n+k,\boldsymbol{\mu}_2},$$

where the sum is taken over all k, $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ such that $\boldsymbol{\mu}_1 r^k = \boldsymbol{\lambda}_1$, $1 \leqslant |\boldsymbol{\mu}_1| \leqslant L_E$, and $\boldsymbol{\mu}_2 r^k = \boldsymbol{\lambda}_2$, so that

$$\Big(\prod A_{n+k} {B_{n+k}}^{|\boldsymbol{\mu}_2|}\Big) \cdot \sigma_{\boldsymbol{\lambda}}^{(n)} \in O_{\boldsymbol{K}},$$

where the product is taken over the same set as above and the cardinality of the set is less than $(L+1)^l$. Hence we have for large n,

$$\begin{aligned}\operatorname{den}\big(\sigma_{\boldsymbol{\lambda}}^{(n)}\big) &\leqslant \prod \kappa^{r^{n+k} + r^{n+k}|\boldsymbol{\mu}_2|} \\ &\leqslant \kappa^{r^n(|\boldsymbol{\lambda}_1| + |\boldsymbol{\lambda}_2|)\cdot(L+1)^l} \\ &= \big(\kappa^{(L+1)^l}\big)^{|\boldsymbol{\lambda}|r^n}.\end{aligned}$$

Next we assume (22). Then $D^{|\boldsymbol{\lambda}|} c_{k\boldsymbol{\lambda}} \in O_{\boldsymbol{K}}$. We have

$$\sigma_{\boldsymbol{\lambda}}^{(n)} = \sum a_{n+k,\boldsymbol{\mu}_1} c_{n+k,\boldsymbol{\mu}_2},$$

where the sum is taken over all k, $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ such that $\boldsymbol{\mu}_1 r^k + \boldsymbol{\mu}_2 r^k = \boldsymbol{\lambda}$, $1 \leqslant |\boldsymbol{\mu}_1| \leqslant L_E$, so that

$$\Big(\prod A_{n+k}\Big) D^{|\boldsymbol{\lambda}|} \sigma_{\boldsymbol{\lambda}}^{(n)} \in O_{\boldsymbol{K}},$$

where the product is taken over the same set as above. Hence we get for large n,

$$\begin{aligned}\operatorname{den}\big(\sigma_{\boldsymbol{\lambda}}^{(n)}\big) &\leqslant \Big(\prod \kappa^{r^{n+k}}\Big) D^{|\boldsymbol{\lambda}|} \\ &\leqslant \kappa^{r^n(r^0 + r^1 + \cdots + r^{[\log_r |\boldsymbol{\lambda}|]})(L+1)^m} D^{|\boldsymbol{\lambda}|} \\ &\leqslant \kappa^{2(L+1)^m r^n |\boldsymbol{\lambda}|} D^{|\boldsymbol{\lambda}|} \\ &= \big(D^{1/r^n} \kappa^{2(L+1)^m}\big)^{|\boldsymbol{\lambda}|r^n}.\end{aligned}$$

The proof is completed. □

Remark 3.1. Lemma 7 is a generalization of Lemma 2 in [6]. In the proof of the Transcendence Criterion in [6], they use the assumption $F_k(x) \in O_{\boldsymbol{K}}[x]$ only in Lemma 2. Hence the Transcendence Criterion is valid when $F_k(x) \in \boldsymbol{K}[x]$ and if there exists a positive integer D such that $DF_k(x) \in O_{\boldsymbol{K}}[x]$ for every k.

That is, we insist that Theorems 7, 8, and 9 in [6] are available in the case of $DF_k(x) \in O_{\boldsymbol{K}}[x]$. We will use these theorems in our proof of Theorem 1.

Lemma 8. *Let $S = \Phi_0(z)$ satisfy* (4). *Then $\{\Phi_n(z)\}_{n\geqslant 0}$ satisfies* (V).

Proof. Let $\deg_z P_i(z;s) \leqslant N$ $(0 \leqslant i \leqslant d)$ and put $I_n = \operatorname{ord}(E(z;\sigma^{(n)}))$. By (8), we have

$$E\big(\Omega_n z;\sigma^{(n)}\big) = \sum_{j=0}^{d} P_j\big(\Omega_n z;\sigma^{(n)}\big)\big(\Phi_n(\Omega_n z)\big)^j = \sum_{j=0}^{d} P_j\big(\Omega_n z;\sigma^{(n)}\big)(S-T_n)^j,$$

and so

$$\begin{aligned} I_n r^n &= \operatorname{ord}\big(E\big(\Omega_n z;\sigma^{(n)}\big)\big) \\ &= \operatorname{ord}\left(\sum_{j=0}^{d} P_j\big(\Omega_n z;\sigma^{(n)}\big)(S-T_n)^j\right) \\ &= \operatorname{ord}\left(\sum_{j=0}^{d} P_j\big(\Omega_n z;\sigma^{(n)}\big)D_n^d(S-T_n)^j\right). \end{aligned}$$

If $P_1(z;\sigma^{(n)}),\dots,P_d(z;\sigma^{(n)})$ are not all zero, we get by (12) and Theorem 3

$$I_n r^n \leqslant c_d\big(Nr^n + dLr^n\big) = c_d(N+dL)r^n.$$

Therefore $I_n \leqslant c_d(N+dL)$, which implies the property (V). □

Consequently, we have

Theorem 9. *Let $S = \Phi_0(z)$ satisfy* (21) *or* (22) *in Lemma* 7 *and* (4), *then $\Phi_0(\boldsymbol{\alpha})$ is a transcendental number.*

The following lemmas will be used in the next section. For $\boldsymbol{a} = (a_1,\dots,a_m)$, $\boldsymbol{b} = (b_1,\dots,b_m) \in \mathbb{N}_0^m$, we denote $\boldsymbol{a} \leqslant \boldsymbol{b}$ if and only if $a_i \leqslant b_i$ for each i.

Lemma 10. (See [16, Lemma 2.6.4].) *If R is a subset of $\mathbb{N}_0^m$. Then there is a finite subset T of R such that for any $\boldsymbol{s} \in R$ there is an element $\boldsymbol{t} \in T$ with $\boldsymbol{t} \leqslant \boldsymbol{s}$.*

Lemma 11. *We assume* (21) *or* (22). *Let M be an infinite subset of* $\mathbb{N}$. *Suppose that there exists* $I \in \mathbb{N}$ *such that* $\operatorname{ord}(\Phi_n(z)) \leqslant I$ *for every* $n \in M$. *Then there exists* γ *with* $0 < \gamma < 1$ *such that* $0 < |\Phi_n(\Omega_n \boldsymbol{\alpha})| < 2\gamma^{r^n}$ *for infinitely many* $n \in M$.

Proof. For $n \in M$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)$, we put

$$\langle \boldsymbol{\mu}_n, \boldsymbol{\eta} \rangle = \min\bigl\{\langle \boldsymbol{\lambda}, \boldsymbol{\eta} \rangle \bigm| \sigma_{\boldsymbol{\lambda}}^{(n)} \neq 0\bigr\},$$

where $\boldsymbol{\mu}_n$ is uniquely determined, since $\eta_1, \ldots, \eta_m$ are linearly independent over $\mathbb{Q}$. We have $\langle \boldsymbol{\mu}_n, \boldsymbol{\eta} \rangle > 0$ as $\sigma_{\mathbf{0}}^{(n)} = 0$. Since $\langle \boldsymbol{\mu}_n, \boldsymbol{\eta} \rangle \leqslant I(\eta_1 + \cdots + \eta_m)$, there exist $\boldsymbol{\mu}$ and an infinite subset M' of M such that $\boldsymbol{\mu}_n = \boldsymbol{\mu}$ for every $n \in M'$. Then we put

$$R = \bigl\{\boldsymbol{\lambda} \in {\mathbb{N}_0}^m \bigm| \langle \boldsymbol{\lambda}, \boldsymbol{\eta} \rangle > \langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle\bigr\}.$$

By Lemma 7, we can prove that, for any $\varepsilon > 0$,

$$\log\bigl|\sigma_{\boldsymbol{\lambda}}^{(n)}(\Omega_n \boldsymbol{\alpha})^{\boldsymbol{\lambda}}\bigr| \leqslant -(1-\varepsilon)\langle \boldsymbol{\lambda}, \boldsymbol{\eta} \rangle r^n,$$

if n is large. Hence, noting Lemma 10, we have for $n \in M'$

$$\begin{aligned}\bigl|\Phi_n(\Omega_n \boldsymbol{\alpha}) - \sigma_{\boldsymbol{\mu}}^{(n)}(\Omega_n \boldsymbol{\alpha})^{\boldsymbol{\mu}}\bigr| &= \Bigl|\sum_{\boldsymbol{\lambda} \in R} \sigma_{\boldsymbol{\lambda}}^{(n)}(\Omega_n \boldsymbol{\alpha})^{\boldsymbol{\lambda}}\Bigr| \\ &\leqslant \sum_{\boldsymbol{\nu} \in T} \sum_{\boldsymbol{\lambda} \geqslant \boldsymbol{\nu}} \bigl|\sigma_{\boldsymbol{\lambda}}^{(n)}(\Omega_n \boldsymbol{\alpha})^{\boldsymbol{\lambda}}\bigr|.\end{aligned}$$

Using Lemma 7 and (6), we get for $\boldsymbol{\nu} \in T$ and $n \in M'$,

$$\begin{aligned}\sum_{\boldsymbol{\lambda} \geqslant \boldsymbol{\nu}} \frac{|\sigma_{\boldsymbol{\lambda}}^{(n)}(\Omega_n \boldsymbol{\alpha})^{\boldsymbol{\lambda}}|}{|\sigma_{\boldsymbol{\mu}}^{(n)}(\Omega_n \boldsymbol{\alpha})^{\boldsymbol{\mu}}|} &\leqslant \sum_{\boldsymbol{\lambda} \geqslant \boldsymbol{\nu}} e^{-(1-\varepsilon)\langle \boldsymbol{\lambda}, \boldsymbol{\eta} \rangle r^n} e^{(2[\boldsymbol{K}:\mathbb{Q}]\varepsilon+1)\langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle r^n} \\ &\leqslant \sum_{\lambda_1 \geqslant \nu_1} \cdots \sum_{\lambda_m \geqslant \nu_m} e^{-(1-\varepsilon)\langle \boldsymbol{\lambda}, \boldsymbol{\eta} \rangle r^n} e^{(2[\boldsymbol{K}:\mathbb{Q}]\varepsilon+1)\langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle r^n} \\ &\leqslant c_1 e^{-(1-\varepsilon)\langle \boldsymbol{\nu}, \boldsymbol{\eta} \rangle r^n} e^{(2[\boldsymbol{K}:\mathbb{Q}]\varepsilon+1)\langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle r^n} \\ &= c_1 e^{(\varepsilon\langle \boldsymbol{\nu}, \boldsymbol{\eta} \rangle + 2[\boldsymbol{K}:\mathbb{Q}]\varepsilon\langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle + \langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle - \langle \boldsymbol{\nu}, \boldsymbol{\eta} \rangle) r^n}.\end{aligned}$$

Since $\langle \boldsymbol{\nu}, \boldsymbol{\eta} \rangle > \langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle$, we can choose $0 < \varepsilon < 1$ such that

$$\varepsilon\bigl(\langle \boldsymbol{\nu}, \boldsymbol{\eta} \rangle + 2[\boldsymbol{K}:\mathbb{Q}]\langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle\bigr) + \bigl(\langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle - \langle \boldsymbol{\nu}, \boldsymbol{\eta} \rangle\bigr) < 0$$

for any $\boldsymbol{\nu} \in T$. Hence $\sum_{\boldsymbol{\lambda} \geqslant \boldsymbol{\nu}} |\sigma_{\boldsymbol{\lambda}}^{(n)}(\Omega_n \boldsymbol{\alpha})^{\boldsymbol{\lambda}}| / |\sigma_{\boldsymbol{\mu}}^{(n)}(\Omega_n \boldsymbol{\alpha})^{\boldsymbol{\mu}}| \to 0$ as $n \in M'$ tends to infinity. Hence we have

$$\left|\frac{\Phi_n(\Omega_n \boldsymbol{\alpha})}{\sigma_{\boldsymbol{\mu}}^{(n)}(\Omega_n \boldsymbol{\alpha})^{\boldsymbol{\mu}}} - 1\right| \to 0$$

as $n \in M'$ tends to infinity. Therefore, we obtain

$$0 < \left|\Phi_n(\Omega_n \boldsymbol{\alpha})\right| < 2e^{-(1-\varepsilon)\langle \boldsymbol{\mu}, \boldsymbol{\eta}\rangle r^n} = 2\gamma^{r^n}$$

for every large $n \in M'$. $\square$

4. Proof of the theorems

We have to prove Theorems 1, 2, and 4.

Proof of Theorem 2. We denote by $\{m(n)\}_{n \geqslant 0}$ the sequence satisfying, for every $n \geqslant 0$ and every k with $m(n) < k < m(n+1)$,

$$E_{m(n)}(z) \neq 0, \qquad E_k(z) = 0.$$

The proof will be divided into two cases.

Case I. Let $\overline{\lim}\{m(n+1) - m(n)\} < +\infty$. Let C be a constant, and independent of n, satisfying $\overline{\lim}\{m(n+1) - m(n)\} \leqslant C$. We apply Theorem 9, proved already in the preceding section. For this we have only to check the condition (4). We may assume $m(0) = 0$, replacing α_i by $\alpha_i^{r^{m(0)}}$. Let $M \geqslant 1$. For $k = 0, 1, \ldots, m - l$, we prove that for $A_0(z), A_1(z) \in \boldsymbol{K}[z_1, \ldots, z_{l+k}]$, not both zero, with $\deg A_0(z), \deg A_1(z) \leqslant M$,

$$\operatorname{ord}\bigl(A_0(z) + A_1(z)\Phi_0(z_1, \ldots, z_{l+k}, 0, \ldots, 0)\bigr) \leqslant \bigl(r^{C+2} + k\bigr)M, \tag{23}$$

by induction on k. Assume $k = 0$. Let $A_0(z), A_1(z) \in \boldsymbol{K}[z_1, \ldots, z_l]$, not both zero, with $\deg A_0(z), \deg A_1(z) \leqslant M$, and

$$A_0(z) + A_1(z)\Phi_0(z_1, \ldots, z_l, 0, \ldots, 0) = f(z).$$

If $A_1(z) = 0$, then $\operatorname{ord} f(z) \leqslant M$. If $A_1(z) \neq 0$, we take $h \in \mathbb{N}$ such that $r^{m(h-1)} \leqslant rM < r^{m(h)}$, noting that $m(0) = 0$. Since $L_E < r$, we have for $1 \leqslant i \leqslant h$, $1 \leqslant j$,

$$\begin{aligned}
\deg\bigl(A_1(z)E_{m(h-i)}(\Omega_{m(h-i)}z)\bigr) &\leqslant M + L_E r^{m(h-1)} \\
&< r^{m(h)-1} + (r-1)r^{m(h-1)} \\
&\leqslant \operatorname{ord}\bigl(A_1(z)E_{m(h)}(\Omega_{m(h)}z)\bigr) \\
&\leqslant \deg\bigl(A_1(z)E_{m(h)}(\Omega_{m(h)}z)\bigr) \\
&\leqslant M + L_E r^{m(h)} \\
&< r^{m(h)} + (r-1)r^{m(h)} \\
&\leqslant \operatorname{ord}\bigl(A_1(z)E_{m(h+j)}(\Omega_{m(h+j)}z)\bigr).
\end{aligned}$$

Since $\deg A_0(z) < \operatorname{ord}(f(z))$ and $F_k(z_1, \ldots, z_l, 0, \ldots, 0) = 1$, noting that $h(z) = P + Q + S$, where the preceding chain of inequalities imply $\deg P < \operatorname{ord} Q$ and $\deg Q < \operatorname{ord} S$; thus $\operatorname{ord}(h(z)) \leqslant \operatorname{ord} Q$, we get

$$\begin{aligned}\operatorname{ord} f(z) &\leqslant \operatorname{ord}\bigl(A_1(z)E_{m(h)}(\Omega_{m(h)}z)\bigr)\\ &\leqslant r^{m(h)+1} \leqslant r^{m(h-1)+C+1} \leqslant r^{C+2}M.\end{aligned}$$

We assume that (23) is true for k. Let $A_0(z), A_1(z) \in \boldsymbol{K}[z_1, \dots, z_{l+k+1}]$, not both zero, with $\deg A_0(z), \deg A_1(z) \leqslant M$. Let $\operatorname{ord}_{z_{l+k+1}}(h(z))$ be $\min\{i \mid \sigma_i(z_1, \dots, z_{l+k}) \neq 0\}$ for $h(z) = \sum_i \sigma_i(z_1, \dots, z_{l+k}) z_{z_{l+k+1}}^i \in \boldsymbol{K}[z_1, \dots, z_{l+k}][z_{l+k+1}]$. If $h(z) = 0$, then we define $\operatorname{ord}_{z_{l+k+1}}(h(z)) = \infty$ for convenience's sake. We put

$$\begin{gathered}A_0(z) + A_1(z)\Phi_0(z_1, \dots, z_{l+k+1}, 0, \dots, 0) = g(z_1, \dots, z_{l+k+1}),\\ s = \min\bigl\{\operatorname{ord}_{z_{l+k+1}} A_0(z), \operatorname{ord}_{z_{l+k+1}} A_1(z)\bigr\},\\ g^*(z_1, \dots, z_{l+k+1}) = z_{l+k+1}^{-s} g(z_1, \dots, z_{l+k+1}),\end{gathered}$$

and

$$B_0(z_1, \dots, z_{l+k+1}) = z_{l+k+1}^{-s} A_0(z), \qquad B_1(z_1, \dots, z_{l+k+1}) = z_{l+k+1}^{-s} A_1(z).$$

Then $B_0(z_1, \dots, z_{l+k}, 0), B_1(z_1, \dots, z_{l+k}, 0)$ are not both zero. By induction hypothesis, we have

$$\begin{aligned}&g^*(z_1, \dots, z_{l+k}, 0)\\ &\quad = B_0(z_1, \dots, z_{l+k}, 0)\\ &\qquad + B_1(z_1, \dots, z_{l+k}, 0)\Phi_0(z_1, \dots, z_{l+k}, 0, \dots, 0)\\ &\quad \leqslant \bigl(r^{C+2} + k\bigr)M.\end{aligned}$$

Hence we get

$$\begin{aligned}\operatorname{ord} g(z_1, \dots, z_{l+k}, z_{l+k+1}) &\leqslant \operatorname{ord} z_{l+k+1}^s g^*(z_1, \dots, z_{l+k}, 0)\\ &\leqslant \bigl(r^{C+2} + k + 1\bigr)M,\end{aligned}$$

which is (23) with $k+1$ in place of k. If $k = m - l$, (23) implies (4), and the transcendence of $\Phi_0(\boldsymbol{\alpha})$ follows from Theorem 9.

Case II. Let $\overline{\lim}\{m(n+1) - m(n)\} = +\infty$. We have

$$\Phi_{m(n+1)}(\Omega_{m(n+1)}\boldsymbol{\alpha}) = \Phi_0(\boldsymbol{\alpha}) - T_{m(n)+1}(\boldsymbol{\alpha}). \tag{24}$$

Since $\deg E_{m(n+1)}(z) \leqslant L_E < r$ and $F_{m(n+1)}(z)$ is independent of $z_1, \dots, z_l$, it cannot interfere the term of smallest order in $E_{m(n+1)}(z)$. Hence we get

$$\operatorname{ord}\bigl(\Phi_{m(n+1)}(z)\bigr) = \operatorname{ord}\bigl(E_{m(n+1)}(z)\bigr) \leqslant L_E = I$$

for ever n. There exists a sequence $\{l(n)\}_{n \geqslant 0}$ of natural numbers such that $\lim\{m(l(n)+1) - m(l(n))\} = +\infty$. We define the set $M = \{m(l(n)+1) \mid n \geqslant 0\}$. By Lemma 11, we have

$$0 < \bigl|\Phi_{m(l(n)+1)}(\Omega_{m(l(n)+1)}\boldsymbol{\alpha})\bigr| \leqslant 2\gamma^{r^{m(l(n)+1)}}, \tag{25}$$

with $\gamma < 1$, for infinitely many n. We assume that $\Phi_0(\boldsymbol{\alpha}) \in \overline{\mathbb{Q}}$. It follows from (7) and Lemma 6, that

$$\left\| \Phi_0(\boldsymbol{\alpha}) - T_{m(l(n))+1}(\boldsymbol{\alpha}) \right\| \leqslant 2c_2 c_3^{r^{m(l(n))+1}}, \tag{26}$$

for every large n. By (24)–(26), and the fundamental inequality, we get

$$\begin{aligned}
&\log 2 + r^{m(l(n)+1)} \log \gamma \\
&\quad \geqslant \log \left| \Phi_{m(l(n)+1)}(\Omega_{m(l(n)+1)}\boldsymbol{\alpha}) \right| \\
&\quad \geqslant -2\big[\boldsymbol{K}\big(\Phi_0(\boldsymbol{\alpha})\big) : \mathbb{Q}\big] \log \left\| \Phi_{m(l(n)+1)}(\Omega_{m(l(n)+1)}\boldsymbol{\alpha}) \right\| \\
&\quad = -2\big[\boldsymbol{K}\big(\Phi_0(\boldsymbol{\alpha})\big) : \mathbb{Q}\big] \log \left\| \Phi_0(\boldsymbol{\alpha}) - T_{m(l(n))+1}(\boldsymbol{\alpha}) \right\| \\
&\quad \geqslant -2\big[\boldsymbol{K}\big(\Phi_0(\boldsymbol{\alpha})\big) : \mathbb{Q}\big]\big(\log 2c_2 + r^{m(l(n))+1} \log c_3\big),
\end{aligned}$$

for infinitely many n; which is a contradiction, if n is large. □

Proof of Theorem 4. There exists $h \in \mathbb{N}$ such that $b_{1n} \neq 0$ and $b_{1n}\rho_1^{r^n} + \cdots + b_{mn}\rho_m^{r^n} \neq 0$ for any $n \geqslant h$. We prove

$$\theta = \sum_{k \geqslant h} \frac{a_k}{b_{1k}\rho_1^{r^k} + \cdots + b_{mk}\rho_m^{r^k}} \notin \overline{\mathbb{Q}}.$$

We put

$$E_k(z) = 0, \quad F_k(z) = 1 \quad (0 \leqslant k < h),$$
$$E_k(z) = {b_{1k}}^{-1} a_k z_1, \quad F_k(z) = 1 + b_{2k}{b_{1k}}^{-1} z_2 + \cdots + b_{mk}{b_{1k}}^{-1} z_m \quad (h \leqslant k),$$

and define

$$\Phi_0(z) = \sum_{k \geqslant 0} \frac{E_k(\Omega_k z)}{F_k(\Omega_k z)}.$$

Noting that $|1/\rho_1|, |\rho_2/\rho_1|, \ldots, |\rho_m/\rho_1|$ are less than 1 and multiplicatively independent, we have $\theta = \Phi_0(1/\rho_1, \rho_2/\rho_1, \ldots, \rho_m/\rho_1) \notin \overline{\mathbb{Q}}$ by Theorem 2. □

Proof of Theorem 1. If $A_2 = 0$, then $\rho_2 = 0$. Therefore ρ_1 is a non-zero integer. If $|\rho_1| = 1$, $\{R_n\}$ is periodic; a contradiction. Hence we have $|\rho_1| \geqslant 2$ and Theorem 4 implies $\theta \notin \overline{\mathbb{Q}}$. We may assume that $A_2 \neq 0$. If $|\rho_1| \leqslant 1$, then $|\rho_1| = |\rho_2| = 1$, since $|\rho_1||\rho_2| = |A_2| \geqslant 1$. This is impossible, since $\{R_n\}$ is non-periodic. Hence $|\rho_1| > 1$. If $|\rho_1| = |\rho_2|$, then $\rho_2 = -\rho_1$, so that

$$\begin{aligned}
R_{cr^k+d} &= \big(g_1 + g_2(-1)^{cr^k+d}\big)\rho_1^{cr^k+d} \\
&= \big(g_1 + g_2(-1)^{cr+d}\big)\rho_1^{cr^k+d}
\end{aligned}$$

for $k \geqslant 1$, and Theorem 4 implies $\theta \notin \overline{\mathbb{Q}}$. Hence we may assume that $|\rho_1| > |\rho_2| > 0$, $|\rho_1| > 1$. If $g_2 = 0$, $\theta \notin \overline{\mathbb{Q}}$ by Theorem 4. If $g_1 = 0$, then $\rho_2 \in \mathbb{Z} \setminus \{0\}$, since $R_n \in \mathbb{Z}$ and ρ_2 is an algebraic integer. As $\{R_n\}$ is not periodic, we have $|\rho_2| > 1$. Thus $\theta \notin \overline{\mathbb{Q}}$ by Theorem 4. Therefore we may assume that

$$|\rho_1| > |\rho_2| > 0, \quad |\rho_1| > 1, \quad \text{and} \quad g_1 g_2 \neq 0.$$

The binary linear recursive sequences (1) with these properties are classified into the following three types:

Case I. $|A_2| = 1$.

Case II. $|A_2| \geqslant 2$ and ρ_1, ρ_2 are multiplicatively dependent.

Case III. ρ_1, ρ_2 are multiplicatively independent.

In cases I and II, we can apply the Transcendence Criterion (see Remark 3.1). So we have only to determine when $\Phi_0(x)$ defined below is a rational function. We can use Theorems 7, 8, and 9 in [6], criterion for rationality of functions such as $\Phi_0(x)$, since they do not use the assumption $F_k(x) \in O_{\boldsymbol{K}}[x]$ in their proofs.

Case I. We have $0 < |\rho_2| < 1 < |\rho_1| = |\rho_2^{-1}|$. There exists $h \in \mathbb{N}$ such that $cr^k + d \geqslant 0$ and $R_{cr^k+d} \neq 0$ for any $k \geqslant h$ where $c, d \in \mathbb{Z}$ with $c \geqslant 1$. We can write

$$\theta = {\sum_{k=0}^{h-1}}' \frac{a_k}{R_{cr^k+d}} + (1/\rho_1)^d g_1^{-1} \Phi_0\big(\rho_1^{-c}\big),$$

with

$$\Phi_0(x) = \sum_{k \geqslant h} \frac{a_k x^{r^k}}{1 \pm (\rho_2/\rho_1)^d g_1^{-1} g_2 x^{2r^k}},$$

where the signature "−" in the last sum is taken if both of c, r are odd and $\rho_1 = -\rho_2^{-1}$.

Assume that

$$\Phi(x) = \sum_{k \geqslant 0} \frac{a_k x^{r^k}}{1 + b x^{2r^k}}, \quad b = \pm(\rho_2/\rho_1)^d g_1^{-1} g_2$$

is a rational function. If $r \geqslant 4$, $\Phi(x)$ is not a rational function, by Theorem 7 in [6]. Also $\Phi(x)$ is not a rational function by Theorem 8 in [6], if $r = 3$. Because the denominator of $\Phi(x)$ does not satisfy $F_n(x) = 1 + \omega^{3^n} x + \omega^{2\cdot 3^n} x^2$ where ω is a root of unity. In the case of $r = 2$, we can apply Theorem 9 in [6]. Then $\Phi(x)$ is a rational function of the form given in the three cases (i), (ii), and (iii) in Theorem 9 in [6]. We easily see that the case (iii) of Theorem 9 is impossible in our case. Also the case (ii) of Theorem 9, since if ω_1, ω_2 are roots of unity, $(\omega_1/\omega_2)^{2^n} = -1$ does not occur for all large n. Therefore, $\Phi(x)$ is in the case (i) in Theorem 9, so that $b = (\rho_2/\rho_1)^d g_1^{-1} g_2 = -1$ and $a_k = a \in \boldsymbol{K}$ for every large k. This is the case (i) in our Theorem 1. If $a_k = 1$ for all $k \geqslant 0$, more precisely, we have

$$\Phi(x) = \sum_{k \geqslant 0} \frac{x^{2^k}}{1 - x^{2^{k+1}}} = \frac{x}{1-x}$$

(see [12]). In our case (i), since $\Phi_0(x)$ can be expressed by $\Phi(x)$ and a rational function over $\boldsymbol{K}[x]$, we get $\theta \in \boldsymbol{K}(\sqrt{\Delta})$.

Case II. We assume that Δ is not a perfect square. There exist integers p and q, not both zero, with $\rho_1^p\rho_2^q = 1$. We take a norm $N_{\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q}}$, then $|A_2|^{p+q} = 1$. Since $|A_2| \geqslant 2$, $p = -q$. It contradicts $|\rho_1| > |\rho_2|$. That is, Δ is a perfect square. (See Theorem 3 in [3] to get another proof.) It leads that ρ_1, ρ_2 are integers since A_1, A_2 are integers. Then there exists an integer $\gamma \geqslant 2$ such that

$$\rho_1 = \pm\gamma^s, \quad \rho_2 = \pm\gamma^{s-t}, \quad s \geqslant t > 0.$$

We may take $s = t = 1$ if $s = t$. For $k \geqslant 1$, we have

$$\begin{aligned} R_{cr^k+d} &= g_1\rho_1^d(\pm 1)^{cr}\gamma^{scr^k} + g_2\rho_2^d(\pm 1)^{cr}\gamma^{(s-t)cr^k} \\ &= g_1'\rho_1^d\gamma^{scr^k} + g_2'\rho_2^d\gamma^{(s-t)cr^k}. \end{aligned}$$

Hence we can write,

$$\theta = {\sum_{k=0}^{h-1}}' \frac{a_k}{R_{cr^k+d}} + {g_1'}^{-1}(1/\rho_1)^d\Phi_0\big(\gamma^{-c}\big),$$

where

$$\Phi_0(x) = \sum_{k\geqslant h} \frac{a_k x^{sr^k}}{1 + {g_1'}^{-1}g_2'(\rho_2/\rho_1)^d x^{tr^k}}.$$

Suppose that $\Phi(x)$ is a rational function

$$\Phi(x) = \sum_{k\geqslant 0} \frac{a_k x^{sr^k}}{1 - bx^{tr^k}} = \frac{P(x)}{Q(x)}, \quad \text{say},$$

where $P(x), Q(x) \in \boldsymbol{K}[x]$ are chosen coprime and $b = -{g_1'}^{-1}g_2'(\rho_2/\rho_1)^d \in \mathbb{Q}$. Let $S_{\boldsymbol{K}}$ denote the set of all the absolute values on $\boldsymbol{K}$. For $|\ |_{\mathfrak{p}} \in S_{\boldsymbol{K}}$, let $\boldsymbol{K}_{\mathfrak{p}}$ be a completion of $\boldsymbol{K}$ with respect to $\mathfrak{p}$ and $C_{\mathfrak{p}}$ a completion of the algebraic closure of $\boldsymbol{K}_{\mathfrak{p}}$.

We prove that $|b|_{\mathfrak{p}} \leqslant 1$ for any $|\ |_{\mathfrak{p}} \in S_{\boldsymbol{K}}$. On the contrary we assume that $|b|_{\mathfrak{p}} > 1$. We put $G_n(x) = \sum_{k\geqslant n+1} a_k x^{sr^k}/(1 - bx^{tr^k})$. Then we have

$$\frac{a_n x^{sr^n}}{1 - bx^{tr^n}} = \frac{P(x)}{Q(x)} - \sum_{k=0}^{n-1} \frac{a_k x^{sr^k}}{1 - bx^{tr^k}} - G_n(x).$$

Multiplying both sides by $D(x) = Q(x)\prod_{k=0}^{n}(1 - bx^{tr^k})$, we have

$$\begin{aligned} & a_n x^{sr^n} Q(x) \prod_{k=0}^{n-1}\big(1 - bx^{tr^k}\big) \\ &\quad = P(x)\prod_{k=0}^{n}\big(1 - bx^{tr^k}\big) - D(x)\sum_{k=0}^{n-1} \frac{a_k x^{sr^k}}{1 - bx^{tr^k}} - D(x)G_n(x). \end{aligned} \tag{27}$$

Let $\kappa > 1$. If $\mathfrak{p}$ is archimedean, $|a_k|_\mathfrak{p} \leqslant \kappa^{r^k}$ for every large k, since $\log \overline{|a_k|} = o(r^k)$. If $\mathfrak{p}$ is non-archimedean,

$$\begin{aligned}|a_k|_\mathfrak{p} &= (N_{\boldsymbol{K}/\mathbb{Q}}\mathfrak{p})^{-v_\mathfrak{p}(a_k)} \leqslant (N_{\boldsymbol{K}/\mathbb{Q}}\mathfrak{p})^{v_\mathfrak{p}(\mathrm{den}(a_k))} \leqslant N_{\boldsymbol{K}/\mathbb{Q}}\big(\mathrm{den}(a_k)\big)\\ &= \big(\mathrm{den}(a_k)\big)^{[\boldsymbol{K}:\mathbb{Q}]} \leqslant \kappa^{r^k},\end{aligned}$$

for every large k, since $\log \mathrm{den}(a_k) = o(r^k)$. We choose κ as $1 < \kappa < |b|_\mathfrak{p}^{s/tr^{n+1}}$. For any $x \in C_\mathfrak{p}$ with $|x|_\mathfrak{p} < |b|_\mathfrak{p}^{-(\frac{1}{tr^{n+1}})}$, we have

$$\sum_{k \geqslant n+1} \frac{|a_k x^{sr^k}|_\mathfrak{p}}{|1 - bx^{tr^k}|_\mathfrak{p}} \leqslant \sum_{k \geqslant n+1} \left(\kappa |b|_\mathfrak{p}^{-(\frac{s}{tr^{n+1}})}\right)^{r^k} \frac{1}{1 - |b|_\mathfrak{p} |x|_\mathfrak{p}^{tr^{n+1}}} < \infty.$$

That is, $G_n(x)$ is absolute convergence for $|x|_\mathfrak{p} < |b|_\mathfrak{p}^{-(\frac{1}{tr^{n+1}})}$. If we substitute $x = b^{-\frac{1}{tr^n}}$ in (27), the right-hand side is equal to zero, and so if $a_n \neq 0$, we have $Q(b^{-\frac{1}{tr^n}}) = 0$. This is a contradiction, since $a_n \neq 0$ for infinitely many n. Therefore $|b|_\mathfrak{p} \leqslant 1$ for any $|\ |_\mathfrak{p} \in S_{\boldsymbol{K}}$. Hence we have $b = \pm 1$, since $b \in \mathbb{Q}$.

If $b = 1$, then

$$\Phi(x) - \sum_{k=0}^{n-1} \frac{a_k x^{sr^k}}{1 - x^{tr^k}} = \sum_{k \geqslant n} \frac{a_k x^{sr^k}}{1 - x^{tr^k}}.$$

Multiplying both sides by $D(x) = Q(x)(1 - x^{tr^{n-1}})$, we have

$$P(x)\big(1 - x^{tr^{n-1}}\big) - D(x) \sum_{k=0}^{n-1} \frac{a_k x^{sr^k}}{1 - x^{tr^k}} = D(x) \sum_{k \geqslant n} \frac{a_k x^{sr^k}}{1 - x^{tr^k}}.$$

The left-hand side is a polynomial of degree less than or equal to $\max\{\deg P(x), \deg Q(x)\} + sr^{n-1}$. On the other hand, if $a_n \neq 0$ the order of the right-hand side is sr^n because $Q(0) \neq 0$. Hence we get

$$sr^n \leqslant \max\big\{\deg P(x), \deg Q(x)\big\} + sr^{n-1},$$

for infinitely many n. This is a contradiction, since $r \geqslant 2$.

If $b = -1$, then

$$\Phi(x) - \sum_{k=0}^{n-1} \frac{a_k x^{sr^k}(1 - x^{tr^k})}{1 - x^{2tr^k}} = \sum_{k \geqslant n} \frac{a_k x^{sr^k}}{1 + x^{tr^k}}.$$

Multiplying both sides by $D(x) = Q(x)(1 - x^{2tr^{n-1}})$, we have

$$P(x)\big(1 - x^{2tr^{n-1}}\big) - D(x) \sum_{k=0}^{n-1} \frac{a_k x^{sr^k}(1 - x^{tr^k})}{1 - x^{2tr^k}} = D(x) \sum_{k \geqslant n} \frac{a_k x^{sr^k}}{1 + x^{tr^k}}.$$

The degree of the left-hand side is less than or equal to $\max\{\deg P(x), \deg Q(x)\} + (s+t)r^{n-1}$. On the other hand, if $a_n \neq 0$ the order of the right-hand side is sr^n because $Q(0) \neq 0$. Hence we have $s = t = 1$ and $r = 2$. Therefore we obtain $\rho_2 = \pm 1$ and $(\rho_2/\rho_1)^d g_1^{-1} g_2 = 1$. Applying Theorem 8 in [6], there exists $N \in \mathbb{N}$ such that $a_n = a2^n$ for every $n \geqslant N$, where $a \in \boldsymbol{K}$ is a constant. This is the case (ii) in Theorem 1. If $a_n = 2^n$ for $n \geqslant 0$, more precisely, we have

$$\Phi(x) = \sum_{k \geqslant 0} \frac{2^k x^{2^k}}{1 + x^{2^k}} = \frac{x}{1-x}$$

(see [5]).

Case III. We have $|\rho_1| > \max\{1, |\rho_2|\}$, where $|\rho_1|, |\rho_2|$ are multiplicatively independent. Hence $\theta \notin \overline{\mathbb{Q}}$ by Theorem 4. □

Acknowledgment

The author thanks Professor Kumiko Nishioka for her kind advice, especially to use all the absolute values on $\boldsymbol{K}$ in the proof of the case II.

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